On the AF Embeddability of Crossed Products of AF Algebras by the Integers

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This paper is concerned with the question of when the crossed product of an AF algebra by an action of $\mathbb{Z}$ is itself AF embeddable. It is well known that quasidiagonality and stable finiteness are hereditary properties. That is, if A and B are C*-algebras with $A \subset B$ and B has either of these properties, then so does A. Since AF algebras enjoy both of these properties we have that quasidiagonality and stable finiteness are geometric obstructions to AF embeddability. For crossed products of AF algebras by $\mathbb{Z}$, these turn out to be the only obstructions.

If A is an AF algebra, then we may easily describe an algebraic obstruction to the AF embeddability of $A \times_\alpha \mathbb{Z}$.

Definition 0.1 If A is an AF algebra and $\alpha \in \mathrm{Aut}(A)$ then we denote by H_α the subgroup of $K_0(A)$ given by all elements of the form $\alpha_*(x) - x$ for $x \in K_0(A)$.

It follows from the Pimsner-Voiculescu six term exact sequence ([PV]) that if A is unital and AF then $K_0(A \times_\alpha \mathbb{Z}) = K_0(A)/H_\alpha$. Now, if B is unital and stably finite (in particular, if B is AF) and $p \in M_n(B)$ is a projection then $[p]$ must be a nonzero element of $K_0(B)$. Thus, if $A \times_\alpha \mathbb{Z}$ embeds into B (or if $A \times_\alpha \mathbb{Z}$ is already stably finite) then every projection in the matrices over $A \times_\alpha \mathbb{Z}$ must give a nonzero element in $K_0(A \times_\alpha \mathbb{Z})$. In particular, since $A \hookrightarrow A \times_\alpha \mathbb{Z}$, and we have observed that $K_0(A \times_\alpha \mathbb{Z}) = \mathbb{K}_{\not\vdash}(\mathbb{A})/\mathbb{H}_\alpha$, we then conclude that $H_\alpha \cap K_0^+(A) = \{0\}$, where $K_0^+(A)$ is the positive cone of $K_0(A)$. Thus, an algebraic obstruction to AF embeddability of a crossed product would be if H_α contained a positive element of $K_0(A)$. It turns out that (so long as A is AF) this is the only algebraic obstruction. Indeed, the main result of this paper is the following.

Theorem 0.2 If A is an AF algebra (not necessarily unital) and $\alpha \in \mathrm{Aut}(A)$ then the following are equivalent.

1. $A \times_\alpha \mathbb{Z}$ is AF embeddable
2. $A \times_\alpha \mathbb{Z}$ is quasidiagonal
3. $A \times_\alpha \mathbb{Z}$ is stably finite
4. $H_\alpha \cap K_0^+(A) = \{0\}$

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are relatively straightforward (whether or not A is unital). In fact, if A is any C*-algebra with a countable approximate unit consisting of projections, then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold. However the remaining implication, (4) $\Rightarrow$ (1), will require some technical machinery (see Section 3 for the details of all the implications). This paper consists of five sections.

In Section 1 we collect a few useful facts and set some of our notation.

In Section 2 we develop the key technical tools. Using the Rohlin property for automorphisms of AF algebras (see Definition 2.1) we will show that certain commutative diagrams at the level of K-theory lift to commutative diagrams on the algebras. In recent years, the Rohlin property has been studied intensively. There are notions of the Rohlin property for more general group actions ([Oc], [Na]) and this notion for actions of $\mathbb{Z}$ has been used in many interesting applications ([Rø], [Ki2], [BKRS], [Co2], [EK], just to name a few). All of the embedding results presented here could be viewed as more applications of the Rohlin property.

In Section 3 we prove a useful K-theoretic characterization of AF embeddability (different than that given in Theorem 0.2) and use this to prove Theorem 0.2.

In Section 4 we present several applications of our results. Among other things, we will prove a natural generalization of a result of Voiculescu that states that if some power of an automorphism of an AF algebra is approximately inner then the corresponding crossed product is AF embeddable (see Theorem 3.6 in [Vo]). We will also recover Pimsner's criterion for AF embeddability of crossed products of $C(X)$ by $\mathbb{Z}$, in the case X is compact, metrizable and totally disconnected.

In Section 5 we show that our constructions can be done in such a way as to yield rationally injective maps on $K_0(A \times_\alpha \mathbb{Z})$ and thus injective maps when $K_0(A \times_\alpha \mathbb{Z})$ is assumed to be torsion free.

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Section 1: Preliminaries

In this section we will recall a few facts and present and easy lemma that will be useful later on.

Throughout this paper, A and B will always denote (not necessarily unital) AF algebras. Following the notation and terminology in [Da], we will let $K_0^+(A)$ and $\Gamma(A)$ be the positive cone and scale, respectively, of $K_0(A)$. We will say that a group homomorphism, θ: $K_0(A) \to K_0(B)$, is a *contraction* if $\theta(\Gamma(A)) \subset \Gamma(B)$. We will say θ is *faithful* if it is contractive and $\Gamma(A) \cap Ker(\theta) = \{0\}$.

Both of the following two facts are essentially contained in Elliot's original classification of AF algebras ([El]). Section IV.4 in [Da] has a very nice treatment of this result. Moreover, the following two facts follow easily from Lemma IV.4.2 and the proof of Elliot's Theorem (Theorem IV.4.3) in [Da].

Fact 1.1 *If θ: $K_0(A) \to K_0(B)$ is a faithful group homomorphism then there exists a *-monomorphism, φ: $A \to B$, with $\varphi_* = \theta$.*

Fact 1.2 *If φ, σ: $A \to B$ are *-homomorphisms, where A is finite dimensional, then $\varphi_* = \sigma_* \Longleftrightarrow \varphi = Adu \circ \sigma$ for some unitary, $u \in B$.*

Recall that when A is unital, $A \times_\alpha \mathbb{Z}$ is defined to be the universal C*-algebra generated by A and a unitary u, subject to the relation $Adu(a) = \alpha(a)$, for all $a \in A$. We will call u the *distinguished* unitary in $A \times_\alpha \mathbb{Z}$. Since one always has the freedom to tensor with other AF algebras when proving AF embeddability, the following lemma will be useful.

Lemma 1.3 If A and B are unital, $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ are given, then there is a natural embedding

$$A \otimes B \times_{\alpha\otimes\beta} \mathbb{Z} \hookrightarrow (A \times_\alpha \mathbb{Z}) \otimes (\mathbb{B} \times_\beta \mathbb{Z})$$

Proof Let $u \in A \times_\alpha \mathbb{Z}$, $v \in B \times_\beta \mathbb{Z}$, and $w \in A \otimes B \times_{\alpha\otimes\beta} \mathbb{Z}$ be the respective distinguished unitaries. Then define a covariant representation by

$$w \longmapsto u \otimes v$$
$$a \otimes b \longmapsto a \otimes b$$

Let $\varphi : A \otimes B \times_{\alpha\otimes\beta} \mathbb{Z} \to (\mathbb{A} \times_\alpha \mathbb{Z}) \otimes (\mathbb{B} \times_\beta \mathbb{Z})$ be the induced *-homomorphism and $D = \mathrm{range}(\varphi)$. To prove that φ is injective (see Thm. 4 in [La]), we must provide automorphisms, $\rho(\xi) \in \mathrm{Aut}(D)$, for all $\xi \in \mathbb{C}$ with $|\xi| = 1$, such that $\rho(\xi)(a \otimes b) = a \otimes b$ and $\rho(\xi)(u \otimes v) = \xi(u \otimes v)$. So, take $\rho_A(\xi) \in \mathrm{Aut}(A \times_\alpha \mathbb{Z})$ such that $\rho_A(\xi)(a) = a$, $\forall\ a \in A$, and $\rho_A(\xi)(u) = \xi u$. Then let $\rho(\xi) = \rho_A(\xi) \otimes id_{B\times_\beta\mathbb{Z}}$ and it is easy to check that $\rho(\xi)(D) = D$ and $\rho(\xi)|_D$ satisfies the desired properties. $\square$

Section 2: The Rohlin Property

We now state the definition of the Rohlin property in the unital case. The appropriate definition in the non-unital case can be found in [EK].

Definition 2.1 If B is unital and $\beta \in \mathrm{Aut}(B)$ then β satisfies the Rohlin property if for every $k \in \mathbb{N}$ there are positive integers $k_1, \ldots, k_m \geq k$ satisfying the following condition: For every finite subset, $\mathcal{F} \subset B$, and every $\epsilon > 0$, there exist projections $e_{i,j},\ i = 1, \ldots, m,\ j = 0, \ldots, k_i - 1$ in B such that

$$\sum_{i=1}^{m} \sum_{j=0}^{k_i-1} e_{i,j} = 1_B$$

$$\|\beta(e_{i,j}) - e_{i,j+1}\| < \epsilon$$

$$\|[x, e_{i,j}]\| < \epsilon$$

for $i = 1, \ldots, m,\ j = 0, \ldots, k_i - 1$ and all $x \in \mathcal{F}$ (where $e_{i,k_i} = e_{i,0}$). For each integer i above, the projections $\{e_{i,j}\}_{j=0}^{k_i-1}$ are called a *Rohlin tower.*

Example 2.2 It follows from Theorem 1.3 in [Ki2] that every UHF algebra admits an automorphism with the Rohlin property. We now give a concrete example of such an automorphism which will be used repeatedly. Let M_n be the $n \times n$ matrices over $\mathbb{C}$ with canonical matrix units $E_{i,j}^{(n)}$. Now let $u_n \in M_n$ be the unitary matrix such that $Adu_n(E_{i,i}^{(n)}) = E_{i+1,i+1}^{(n)}$ (with addition modulo n). Now consider the Universal UHF algebra $\mathcal{U} = \otimes_{\backslash \geq \infty} \mathcal{M}_\backslash$ and the automorphism $\sigma = \otimes_{n \geq 1} Adu_n \in \mathrm{Aut}(\mathcal{U})$. We claim that σ has the Rohlin property. It is important to note that the integers $k_1, \ldots, k_m$ in the definition of the Rohlin property must be fixed and must work for all finite subsets and for all choices of ϵ. We will show that for this particular example, one may always take $m = 1$ and $k_1 = k$. So, let $\mathcal{F} \subset \mathcal{U}$ be a finite subset and $\epsilon > 0$ be given. Now, take some large $m \in \mathbb{N}$ such that $\mathcal{F}$ is nearly contained (within $\epsilon/2$ will suffice) in $\otimes_{n=1}^{m} M_n \subset \mathcal{U}$. Now take $m' > m$ such that k divides m', say $m' = sk$. Then let $e_j = \sum_{t=0}^{s-1} E_{j+tk,j+tk}^{(m')} \in \mathcal{U}$ (where we have identified $M_{m'}$ with it's unital image in $\mathcal{U}$), for $1 \leq j \leq k$. Evidently we have $\sigma(e_j) = e_{j+1}$, $\sum_{j=1}^{k} e_j = 1_{\mathcal{U}}$ and the e_j's nearly commute with $\mathcal{F}$ since they commute with $\otimes_{n=1}^{m} M_n$ by construction.

We would also like to point out that $\mathcal{U} \times_\sigma \mathbb{Z}$ is AF embeddable (and actually embeds back into $\mathcal{U}$) by Lemma 2.8 in [Vo]. In fact every crossed product of a UHF algebra (by $\mathbb{Z}$) is AF embeddable since every automorphism of a UHF algebra is approximately inner (cf. Theorem 3.6 in [Vo]). However, this particular example is also limit periodic (see Definition 5.1) and thus we do not need the full power of Theorem 3.6 in [Vo] to deduce AF embeddability.

Remark 2.3 If $B_0 \subset B$ is a unital, finite dimensional subalgebra and $\Psi : B \to B_0' \cap B$ is a conditional expectation then it is readily verified that $\|\Psi(b) - b\|$ is small whenever b almost commutes with the set of matrix units from which Ψ is constructed. Thus, if the finite subset, $\mathcal{F}$, in the definition of the Rohlin property is taken to be the matrix units of B_0 then we have that the projections, $\{e_{i,j}\}$, are nearly contained in $B_0' \cap B$ and so we can assume without loss of generality that $\{e_{i,j}\} \subset B_0' \cap B$. (see Lemma III.3.1 in [Da]). More generally, if $\mathcal{F}$ is contained in a finite dimensional subalgebra then the Rohlin towers may be chosen to commute with $\mathcal{F}$.

Throughout this section we will mainly be concerned with unital algebras. However, this is only out of convenience and it should be noted that all of the results in this section have analogues in the non-unital case (cf. [EK]).

We now state a theorem due to Evans and Kishimoto that illustrates the usefulness of the Rohlin property when dealing with crossed products. Actually, Theorem 4.1 in [EK] is much stronger than the following.

Theorem 2.4(Evans and Kishimoto) If A is unital, $\alpha, \beta \in \mathrm{Aut}(A)$, $\alpha_* = \beta_*$ on $K_0(A)$, and both α and β satisfy the Rohlin property, then $A \times_\alpha \mathbb{Z} \cong \mathbb{A} \times_\beta \mathbb{Z}$.

Remark 2.5 The following corollary follows immediately from condition (4) of Theorem 0.2 (even without the assumption that A is unital). However, it is easily verified that if α has the Rohlin property then $\alpha \otimes \beta$ does also. Thus using Theorem 2.4 and Lemma 1.3 we may easily prove homotopy invariance of AF embeddability.

Corollary 2.6 If A is unital and $\alpha, \beta \in \mathrm{Aut}(A)$ are homotopic then $A \times_\alpha \mathbb{Z}$ is AF embeddable $\Longleftrightarrow A \times_\beta \mathbb{Z}$ is AF embeddable.

Proof Let $\mathcal{U}$ be the Universal UHF algebra and $\sigma \in \mathrm{Aut}(\mathcal{U})$ be the automorphism with the Rohlin property described in Example 2.2. Then $\alpha \otimes \sigma$ is homotopic to $\beta \otimes \sigma$. Hence, from Theorem 2.4 we have

$$A \otimes \mathcal{U} \times_{\alpha \otimes \sigma} \mathbb{Z} \cong \mathbb{A} \otimes \mathcal{U} \times_{\beta \otimes \sigma} \mathbb{Z}$$

The conclusion now follows from Lemma 1.3 since the crossed product, $\mathcal{U} \times_\sigma \mathbb{Z}$ is AF embeddable. $\Box$

The following stabilization lemma was essentially proved in [EK] (without the assumption of a unit). Our formulation is somewhat different, but our proof is just a more detailed version of that given in [EK].

Lemma 2.7 Let B be unital , $\beta \in \mathrm{Aut}(B)$ have the Rohlin property and $k_1 \geq k_2 \geq \ldots \geq k_m \geq k$ be integers satisfying the definition of the Rohlin property. If $B_0 \subset B_1 \subset B_2$ are finite dimensional subalgebras of B with $1_B \in B_i$, $i = 0, 1, 2$ and

$$\beta^{-j}(B_i) \subset B_{i+1}, \ \text{ for } \ i = 0, 1 \ \text{ and } 0 \leq j \leq k_1$$

then for every unitary, $u \in B_2' \cap B$, there exists a unitary, $v \in B_0' \cap B$ such that $\|u - v\beta(v^*)\| \leq 4/(k-1)$.

Proof (cf. [EK], Lemma 3.2) Let $u \in B_2' \cap B$ be given. Then define

$$\tilde{u}_0 = 1_B, \ \tilde{u}_1 = u, \ \tilde{u}_j = u\beta(u) \cdots \beta^{j-1}(u), \quad \text{for } j \geq 2$$

Notice that our hypotheses imply that for $0 \leq j \leq k_1$ we have $\tilde{u}_j \in B_1' \cap B$ and for every unitary $w \in B_1' \cap B$ we have that $\beta^j(w) \in B_0' \cap B$, $j = 0, \ldots, k_1$. Now, since $B_1' \cap B$ is a unital AF subalgebra of B (cf. Exercise 7.7.5 in [Bl]), we can find a unital, finite dimensional subalgebra, $B_3 \subset B_1' \cap B$, such that there exist unitaries, $u_i \in B_3$ with $\|u_i - \tilde{u}_i\| < \epsilon$, for some small $\epsilon > 0$. Then let $w_{k_i}^{(t)}, i = 1, \ldots, m$, be paths of unitaries in B_3 such that

$$w_{k_i}^{(0)} = 1_B \qquad , \qquad w_{k_i}^{(1)} = u_{k_i}$$
$$\|w_{k_i}^{(s)} - w_{k_i}^{(t)}\| \leq \pi|s - t| \ , \ s, t \in [0, 1]$$

We have already observed that $\beta^j(w_{k_i}^{(t)}) \in B_0' \cap B$ for $0 \leq j \leq k_1$. So, (by Remark 2.3) take a set of Rohlin towers, $\{e_{i,j}\} \subset B_0' \cap B$. We may also assume without loss of generality that the Rohlin towers approximately commute (with error also bounded by ϵ) with the $\tilde{u}_j$'s, $\beta(\tilde{u}_j)$'s and $\beta^j(w_{k_i}^{(1-\frac{j}{k_i-1})})$'s below. So we define

$$v' = \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} \tilde{u}_j \beta^j (w_{k_i}^{(1-\frac{j}{k_i-1})}) e_{i,j}$$

Now, v' is not a unitary. However, it is easy to verify that $\|v'v'^* - 1_B\| < \epsilon C$ and $\|v'^*v' - 1_B\| < \epsilon C$, where C is a constant depending only on $k_1, \dots, k_m$. For the remainder of the proof, C will always denote a constant depending only on $k_1, \dots, k_m$. We make no effort to keep track of the best constants as this detail is not needed for the proof. So we have that v' is invertible and hence the unitary, v, in it's polar decomposition actually lives in $B_0' \cap B$ (since $v' \in B_0' \cap B$ also). However, the same estimates imply that v' is also close to this unitary since $|v'|$ will be close to 1_B. More precisely, if we write $v' = v|v'|$ then from elementary spectral theory we have $\|v - v'\| < 1 - \sqrt{1 + \epsilon C}$.

Notice that the definition of the Rohlin property implies the following estimates.

$$\|\beta(e_{i,j})e_{i,j+1} - e_{i,j+1}\| \leq \epsilon$$
$$\|\beta(e_{i,j})e_{k,l}\| \leq \epsilon, \text{ unless } k = i \text{ and } l = j+1$$

Thus a straightforward calculation yields

$$\|v'\beta(v'^*) - \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} \tilde{u}_j \beta^j (w_{k_i}^{(1-\frac{j}{k_i-1})}) e_{i,j} \beta^j (w_{k_i}^{(1-\frac{j-1}{k_i-1})*}) \beta(\tilde{u}^*_{j-1})\| \leq \epsilon C$$

where for each i, the $j = 0$ term in the sum above is actually given by

$$\tilde{u}_0 u_{k_i} e_{i,0} \beta^{k_i}(w_{k_i}^{(1-\frac{k_i-1}{k_i-1})*}) \beta(\tilde{u}^*_{k_i-1}) = u_{k_i} e_{i,0} \beta(\tilde{u}^*_{k_i-1}) \approx \tilde{u}_{k_i} e_{i,0} \beta(\tilde{u}^*_{k_i-1})$$

The following observations, will be needed to get the desired estimate.

a) $\|e_{i,j}\beta^j(w_{k_i}^{(1-\frac{j-1}{k_i-1})*})\beta(\tilde{u}^*_{j-1}) - \beta^j(w_{k_i}^{(1-\frac{j-1}{k_i-1})*})\beta(\tilde{u}^*_{j-1})e_{i,j}\| \leq 2\epsilon$

b) $\|\beta^j(w_{k_i}^{(1-\frac{j}{k_i-1})})\beta^j(w_{k_i}^{(1-\frac{j-1}{k_i-1})*}) - 1_B\| \leq \frac{\pi}{k_i-1}$

c) $\tilde{u}_j \beta(\tilde{u}^*_{j-1}) = u$ for $j \geq 1$

Using the estimates above, we have the following approximations.

$$\begin{aligned}
v'\beta(v'^*) \quad &\overset{\epsilon C}{\approx} \quad \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} \tilde{u}_j \beta^j (w_{k_i}^{(1-\frac{j}{k_i-1})}) e_{i,j} \beta^j (w_{k_i}^{(1-\frac{j-1}{k_i-1})*}) \beta(\tilde{u}^*_{j-1}) \\
&\overset{\epsilon C}{\approx} \quad \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} \tilde{u}_j \beta^j (w_{k_i}^{(1-\frac{j}{k_i-1})}) \beta^j (w_{k_i}^{(1-\frac{j-1}{k_i-1})*}) \beta(\tilde{u}^*_{j-1}) e_{i,j} && \textit{by} \quad (a) \\
&\overset{\frac{\pi}{k-1}+\epsilon C}{\approx} \quad \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} \tilde{u}_j \beta(\tilde{u}^*_{j-1}) e_{i,j} && \textit{by} \quad (b) \\
&\overset{\epsilon C}{\approx} \quad u && \textit{by} \quad (c).
\end{aligned}$$

Recall that the only thing keeping the last approximation from being an equality is the fact that for each i, the $j = 0$ term above is actually given by $u_{k_i}\beta(\tilde{u}^*_{k_i-1})e_{i,0} \overset{\epsilon}{\approx} \tilde{u}_{k_i}\beta(\tilde{u}^*_{k_i-1})e_{i,0} = ue_{i,0}$. Finally, combining all of these estimates we have that

$\|u - v'\beta(v'^*)\| < \frac{\pi}{k-1} + \epsilon C$ and hence $\|u - v\beta(v^*)\| < \frac{\pi}{k-1} + \epsilon C + 2(1 - \sqrt{1+\epsilon C})$, where we have the liberty to take ϵ as small as we like. $\square$

We now turn to the main tool of this paper. The following proposition is the key to Proposition 3.1, which in turn is the key to the rest of the embedding results presented here.

Proposition 2.8 Let A, B be unital, $\alpha \in \mathrm{Aut}(A)$, $\beta \in \mathrm{Aut}(B)$ and $\varphi : A \to B$ a unital, *-monomorphism. Assume also that β satisfies the Rohlin property and $\beta_* \circ \varphi_* = \varphi_* \circ \alpha_*$, i.e. that we have commutativity in the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\varphi_*} & K_0(B) \\ \alpha_* \downarrow & & \downarrow \beta_* \\ K_0(A) & \xrightarrow{\varphi_*} & K_0(B) \end{array}$$

Then there exist unitaries, $v \in A, u \in B$, and a unital *-monomorphism, $\varphi' : A \to B$, with $\varphi'_* = \varphi_*$ and commutativity in

$$\begin{array}{ccc} A & \xrightarrow{\varphi'} & B \\ Adv \circ \alpha \downarrow & & \downarrow Adu \circ \beta \\ A & \xrightarrow{\varphi'} & B \end{array}$$

Proof Let $\{A_i\}_{i=0}^{\infty}$ be an increasing nest of finite dimensional subalgebras whose union is dense in A and $A_0 = \mathbb{C}1_A$. Then $\{\alpha(A_i)\}$ is a nest with the same properties and thus we can find a unitary $v \in A$ (with v as close to 1_A as we like) such that (see Theorem III.3.5 in [Da])

$$Adv \circ \alpha(\bigcup A_i) = \bigcup A_i$$

To ease our notation somewhat, we will henceforth denote $Adv \circ \alpha$ also by α (and just remember that the v in the statement of the proposition has already been chosen).

Now, let $m_i = 2^{i+3} + 1$ and applying the definition of the Rohlin property to m_i we get a finite set of integers, $\{k_1^{(i)}, \ldots, k_{s_i}^{(i)}\}$, with $k_t^{(i)} \geq m_i$ for $1 \leq t \leq s_i$. Then let $m'_i = max\{k_1^{(i)}, \ldots, k_{s_i}^{(i)}\}$.

Since we have arranged that $\bigcup \alpha(A_i) = \bigcup A_i$, by passing to a subsequence we may further assume that for $i \geq 2$ we have that

$$\alpha^{-j}(A_{i-1}) \subset A_i,\ 0 \leq j \leq m'_i$$
$$\alpha^{-j}(A_{i-2}) \subset A_{i-1},\ 0 \leq j \leq m'_i$$

We will now inductively construct sequences of unitaries, $u_i \in B$, automorphisms, $\beta_i \in \mathrm{Aut}(B)$, and unital *-monomorphisms, $\varphi_i : A \to B$, with the following properties

1. $\|u_i - 1_B\| \leq 1/2^i$
2. $\beta_{i+1} = \mathrm{Adu}_{i+1} \circ \beta_i$
3. $\varphi_{i*} = \varphi_*$
4. $\varphi_{i+1}|_{\alpha(A_{i-2})} = \varphi_i|_{\alpha(A_{i-2})}$ where $\mathbb{C}1_A = A_0 = A_{-1} = A_{-2}$
5. $\beta_i \circ \varphi_i|_{A_i} = \varphi_i \circ \alpha|_{A_i}$

where $u_0 = 1_B,\ \varphi_0 = \varphi$ and $\beta_0 = \beta$. Note that each β_i will also have the Rohlin property. So, assume that we have found the desired $u_i,\ \beta_i,\ \varphi_i$ for $0 \leq i \leq n$ and we will show how to construct $u_{n+1},\ \beta_{n+1}$ and φ_{n+1}. By construction, we have

$\beta_{n*} \circ \varphi_{n*} = \varphi_{n*} \circ \alpha_*$ and thus restricting to A_{n+1} we have that $\beta_{n*} \circ \varphi_{n*}$ and $\varphi_{n*} \circ \alpha_*$ agree as maps from $K_0(A_{n+1}) \to K_0(B)$. Thus, (by Fact 1.2) we can find a unitary $w_{n+1} \in B$ such that

$$Adw_{n+1} \circ \beta_n \circ \varphi_n|_{A_{n+1}} = \varphi_n \circ \alpha|_{A_{n+1}}$$

But, by assumption, we already have

$$\beta_n \circ \varphi_n|_{A_n} = \varphi_n \circ \alpha|_{A_n}$$

and hence $w_{n+1} \in \varphi_n(\alpha(A_n))' \cap B$. Now, since we have control of the iterates of A_{n-2} and A_{n-1} (under α) we claim that

$$\beta_n^{-j}(\varphi_n(\alpha(A_{n-1}))) \subset \varphi_n(\alpha(A_n)) \quad , \quad 0 \leq j \leq m'_n$$
$$\beta_n^{-j}(\varphi_n(\alpha(A_{n-2}))) \subset \varphi_n(\alpha(A_{n-1})) \quad , \quad 0 \leq j \leq m'_n$$

To see this we first note that since $\alpha^{-1}(A_{n-1}) \subset A_n$, (5) implies

$$\beta_n^{-1}(\varphi_n(\alpha(A_{n-1}))) = \varphi_n(A_{n-1}) \subset \varphi(\alpha(A_n))$$

Similarly we have

$$\begin{aligned} \beta_n^{-2}(\varphi_n(\alpha(A_{n-1}))) &= \beta^{-1}(\beta^{-1}(\varphi_n(\alpha(A_{n-1})))) \\ &= \beta^{-1}(\varphi_n(A_{n-1})) \\ &= \beta^{-1}(\varphi_n(\alpha(\alpha^{-1}(A_{n-1})))) \\ &= \varphi_n(\alpha^{-1}(A_{n-1})) \quad (\textit{since} \ \ \alpha^{-1}(A_{n-1}) \subset A_n) \end{aligned}$$

but, $\alpha^{-2}(A_{n-1}) \subset A_n$ implies that $\varphi_n(\alpha^{-1}(A_{n-1})) \subset \varphi_n(\alpha(A_n))$. By repeating the above argument, one can show that $\beta_n^{-j}(\varphi_n(\alpha(A_{n-1}))) = \varphi_n(\alpha^{-j+1}(A_{n-1}))$ for $0 \leq j \leq m'_n$. But then $\alpha^{-j}(A_{n-1}) \subset A_n$ implies that $\varphi_n(\alpha^{-j+1}(A_{n-1})) \subset \varphi_n(\alpha(A_n))$, for $0 \leq j \leq m'_n$. The same argument works for the iterates of $\varphi_n(\alpha(A_{n-2}))$ as well.

Thus, by Lemma 2.7, we may take a unitary, $v_{n+1} \in \varphi_n(\alpha(A_{n-2}))' \cap B$ such that $\|w_{n+1} - v_{n+1}\beta_n(v_{n+1}^*)\| \leq \frac{4}{m_n - 1} = \frac{1}{2^{n+1}}$. So, we define

$$\varphi_{n+1} = Adv_{n+1}^* \circ \varphi_n,$$
$$u_{n+1} = v_{n+1}^* w_{n+1} \beta_n(v_{n+1})$$
$$\beta_{n+1} = Adu_{n+1} \circ \beta_n$$

It is now easy to check that we have satisfied all of the required properties.

The proof of the proposition is now complete as it is easy to see that $u = \lim_n u_n \cdots u_1$ is a well defined unitary in B. Also, it is easy to see (by condition 4) that the φ_i's converge (in the point-norm topology) to a unital *-monomorphism, φ', and clearly $Adu \circ \beta \circ \varphi' = \varphi' \circ \alpha$. □

Remark 2.9 It would be of independent interest to know if Proposition 2.6 can be proved without the assumption that β satisfy the Rohlin property. However, the author was unable to either prove or provide a counterexample to such a claim.

Corollary 2.10 Under the assumptions of Proposition 2.8 we also have that $A \times_\alpha \mathbb{Z}$ embeds into $B \times_\beta \mathbb{Z}$.

Proof Essentially what we have done in Proposition 2.8 is embedded A into B in such a way that the automorphism $Adv \circ \alpha$ extends to an automorphism (namely, $Adu \circ \beta$) of B. In general it is true that if C, D are C*-algebras with $C \subset D$, $\gamma \in \mathrm{Aut}(D)$ and $\gamma(C) = C$ then there is a natural inclusion

$$C \times_{\gamma|_C} \mathbb{Z} \hookrightarrow \mathbb{D} \times_\gamma \mathbb{Z}.$$

Thus the conclusion follows from the isomorphisms

$$A \times_\alpha \mathbb{Z} \cong \mathbb{A} \times_{\mathbb{A}\gtrapprox\circ\alpha} \mathbb{Z}, \quad \mathbb{B} \times_\beta \mathbb{Z} \cong \mathbb{B} \times_{\mathbb{A}\approxeq\circ\beta} \mathbb{Z}$$

and the fact that $Adu \circ \beta$ is an extension of $Adv \circ \alpha$ under the embedding φ'. $\Box$

Section 3: Characterizing AF Embeddability

If $\varphi : A \to B$ is a *-homomorphism, we will denote by $\tilde{A}$ and $\tilde{B}$ the C*-algebras obtained by adjoining a (possibly new) unit and we will let $\tilde{\varphi}$ be the (unique) unital extension of φ.

Proposition 3.1 Let A and B be given (not necessarily unital), $\alpha \in \mathrm{Aut}(A), \beta \in \mathrm{Aut}(B)$ and $\varphi : A \to B$ be a *-monomorphism. Further assume that $\beta_* \circ \varphi_* = \varphi_* \circ \alpha_*$ and that $\tilde{B} \times_{\tilde{\beta}} \mathbb{Z}$ is AF embeddable. Then $A \times_\alpha \mathbb{Z}$ is also AF embeddable.

Proof Note that after unitizing everything we still have $\tilde{\beta}_* \circ \tilde{\varphi}_* = \tilde{\varphi}_* \circ \tilde{\alpha}_*$. Let $\mathcal{U}$ be the universal UHF algebra and $\sigma \in \mathrm{Aut}(\mathcal{U})$ be the automorphism (with the Rohlin property) defined in Example 2.2. Now we let $\tilde{\beta}' = \tilde{\beta} \otimes \sigma$ and let $\tilde{\varphi}' : A \to \tilde{B} \otimes \mathcal{U}$ be given by $\tilde{\varphi}'(a) = \tilde{\varphi}(a) \otimes 1_{\mathcal{U}}$. Then $\tilde{\beta}'$ satisfies the Rohlin property and it is easy to check that we still have commutativity at the level of K-theory, i.e. $\tilde{\beta}'_* \circ \tilde{\varphi}'_* = \tilde{\varphi}'_* \circ \tilde{\alpha}_*$. Now, it is always true that

$$A \times_\alpha \mathbb{Z} \hookrightarrow \tilde{\mathrm{A}} \times_{\tilde{\alpha}} \mathbb{Z}$$

and by Corollary 2.10 we have that

$$\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z} \hookrightarrow \tilde{B} \otimes \mathcal{U} \times_{\tilde{\beta}'} \mathbb{Z}.$$

But then by Lemma 1.3 we also have that

$$\tilde{B} \otimes \mathcal{U} \times_{\tilde{\beta}'} \mathbb{Z} \hookrightarrow (\tilde{B} \times_{\tilde{\beta}} \mathbb{Z}) \otimes (\mathcal{U} \times_\sigma \mathbb{Z})$$

where each of the algebras on the right hand side are AF embeddable. $\Box$

Corollary 3.2 If $\alpha \in \mathrm{Aut}(A)$ and $\theta : K_0(A) \to G$ is a positive group homomorphism where G is a dimension group (cf. [Ef]), $Ker(\theta) \cap \Gamma(A) = \{0\}$ and $H_\alpha \subset Ker(\theta)$ then $A \times_\alpha \mathbb{Z}$ is AF embeddable.

Proof Since there exists an AF algebra, B, with $K_0(B) = G$ and $\Gamma(B) = G^+$, we have that θ is a faithful homomorphism and thus by Fact 1.1 we may find a *-monomorphism, $\varphi : A \to B$, with $\varphi_* = \theta$. But we may now take $\beta \in \mathrm{Aut}(B)$ to be the identity and the conclusion follows from Proposition 3.1. $\Box$

Remark 3.3 The hypotheses of Corollary 3.2 are easily seen to be necessary as well. The proof of this necessity is essentially contained in the introduction.

We nearly have the necessary tools to prove Theorem 0.2 now. However, for the implication (4) $\Rightarrow$ (1) we need to provide a candidate AF algebra for the desired embedding of $A \times_\alpha \mathbb{Z}$. The following key lemma due to Spielberg will give us our candidate. We will not prove this lemma (see Lemma 1.14 in [Sp]) but would like to point out that it depends on the Effros-Handelman-Shen Theorem ([EHS]).

Lemma 3.4(Spielberg) If G is a dimension group and H is a subgroup of G with $H \cap G^+ = \{0\}$, then there exists a positive group homomorphism $\theta : G \to G'$ (where G' is also a dimension group) with

1. $H \subset Ker(\theta)$

2. $Ker(\theta) \cap G^+ = \{0\}$

Proof of Theorem 0.2

(1) $\Rightarrow$ (2) is obvious (whether or not A is unital).

(2) $\Rightarrow$ (3) If A is unital then so is $A \times_\alpha \mathbb{Z}$ and thus quasidiagonality implies that every isometry in $A \times_\alpha \mathbb{Z}$ is actually a unitary. But this implies that the identity of $A \times_\alpha \mathbb{Z}$ is not equivalent to any proper subprojection and hence $A \times_\alpha \mathbb{Z}$ is finite. Since matrix algebras over quasidiagonal algebras are again quasidiagonal, the same argument shows that $A \times_\alpha \mathbb{Z}$ is stably finite.

If A is non-unital and $A \times_\alpha \mathbb{Z}$ is quasidiagonal then so is $\widetilde{A \times_\alpha \mathbb{Z}}$. Thus by the above argument we have that $\widetilde{A \times_\alpha \mathbb{Z}}$ is stably finite and thus $A \times_\alpha \mathbb{Z}$ inherits this property also.

(3) $\Rightarrow$ (4) We have already shown this in the introduction when A is unital. However, the following is a very elementary argument which does not depend on the Pimsner-Voiculescu six term exact sequence and which actually holds for much general algebras than just AF algebras.

Since A has a countable approximate unit consisting of projections, we do not need to pass to the unitization of A when computing $K_0(A)$ (see Proposition 5.5.5 in [Bl]). Now, assume that $H_\alpha \cap K_0^+(A) \supsetneqq \{0\}$, i.e. there is some $x \in K_0(A)$ and some projection, $r \in M_n(A)$ such that $\alpha_*(x) - x = [r] \neq 0$. Now, write $x = [p] - [q]$, where (without loss of generality) $p,\ q \in M_n(A)$ are projections. We will not keep track of the sizes of matrices that we are dealing with since this is not important for our argument. So, rewriting the equation $\alpha_*(x) - x = [r]$ we get

$$[\alpha(p)] + [q] = [p] + [\alpha(q)] + [r]$$

Now, since AF algebras enjoy cancellation, we can find a partial isometry (in the matrices over A), v, with support projection $diag(p,\ \alpha(q),\ r)$ and range projection $diag(\alpha(p),\ q,\ 0)$. If we now move to $A \times_\alpha \mathbb{Z}$ and let u be the distinguished unitary (i.e. $uau^* = \alpha(a), \forall\ a \in A$) then it is easy to verify that $diag(pu^*,\ uq,\ 0)$ is a partial isometry with support projection $diag(\alpha(p),\ q,\ 0)$ and range projection $diag(p,\ \alpha(q),\ 0)$. Thus multiplying these two partial isometries we get a partial isometry (in the matrices over $A \times_\alpha \mathbb{Z}$) from $diag(p,\ \alpha(q),\ r)$ to the proper subprojection $diag(p,\ \alpha(q),\ 0)$ and thus $A \times_\alpha \mathbb{Z}$ is not stably finite.

(4) $\Rightarrow$ (1) Letting $H = H_\alpha$ in Lemma 3.4, this implication now follows from Corollary 3.2. $\Box$

Remark 3.5 Note that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold for any C*-algebra, A, while the only property of AF algebras that we really used in the implication (3) $\Rightarrow$ (4) was the fact that one need not pass to the unitization of A when computing $K_0(A)$ (i.e. cancellation is not necessary in the argument above).

Section 4: Applications

We now introduce a simple K-theoretical condition which is sufficient to ensure AF embeddability of the corresponding crossed product. This condition gives a generalization of all the embedding theorems in [Vo] as all of the automorphisms considered there satisfy the following.

Definition 4.1 If $\alpha \in \mathrm{Aut}(A)$ then α satisfies the *finite orbit property*, (FOP), if for every $x \in K_0(A)$ there exists an integer, $0 \neq n \in \mathbb{N}$, such that $\alpha_*^n(x) = x$.

Theorem 4.2 If A (not necessarily unital) is given, $\alpha \in \mathrm{Aut}(A)$ and α satisfies (FOP) then $A \times_\alpha \mathbb{Z}$ is AF embeddable.

Proof We will show the contrapositive. If $A \times_\alpha \mathbb{Z}$ is not AF embeddable then $H_\alpha \cap K_0^+(A) \neq \{0\}$. So, let $x \in K_0(A)$ be chosen so that $0 \neq \alpha_*(x) - x \in K_0^+(A)$.

Note that since α_* is an isomorphism and $K_0^+(A) \cap (-K_0^+(A)) = 0$ we have the following two facts.

$i) If\ 0 \neq [p] \in K_0^+(A)\ then\ 0 \neq \alpha_*([p]) = [\alpha(p)] \in K_0^+(A)$

$ii) If\ 0 \neq [p], [q] \in K_0^+(A)\ then\ 0 \neq [p] + [q] \in K_0^+(A)$

Thus we have that $0 \neq \alpha_*^2(x) - \alpha_*(x)$ and hence $0 \neq (\alpha_*^2(x) - \alpha_*(x)) + (\alpha_*(x) - x) = \alpha_*^2(x) - x \in K_0^+(A)$. Arguing similarly we have that $0 \neq \alpha_*^j(x) - x$, for all nonzero $j \in \mathbb{N}$. Thus α does not satisfy (FOP). $\Box$

Remark 4.3 Recall that Theorem 3.6 in [Vo] states that if there exists a nonzero integer, n, such that $\alpha_*^n = id_{A*}$ then the corresponding crossed product is AF embeddable. The following example shows that Theorem 4.2 is a generalization of this result. Let $X = \{1, 1/2, 1/3, 1/4, \ldots\} \cup \{0\}$ and let $\varphi : X \to X$ be the homeomorphism which leaves 0 fixed, interchanges 1 and 1/2, cyclicly permutes 1/3, 1/4 and 1/5, cyclicly permutes 1/6, 1/7, 1/8 and 1/9 and so on (with increasing lengths of cycles). Then it is easy to see that φ satisfies (FOP), but there is no power of φ which is approximately inner since the only inner automorphism is the identity map.

It is not hard to show that (FOP) is not a necessary condition for AF embeddability (see Remark 4.8). However, automorphisms satisfying (FOP) do admit a nice characterization. It should be pointed out that the following proposition will not be needed in the remainder of this paper.

Note that α satisfies (FOP) if and only if $\tilde{\alpha}$ satisfies (FOP) and thus there is no harm in passing to unitizations when dealing with this property.

Abusing notation a little, we will also denote by α_* the image of an automorphism in the quotient group $\mathrm{Aut}(A)_* = \mathrm{Aut}(A)/\overline{Inn(A)}$. We now show that (FOP) characterizes all the well behaved automorphisms in $\mathrm{Aut}(A)_*$.

Proposition 4.4 Let A be unital and $\alpha \in \mathrm{Aut}(A)$. Then the following are equivalent:

1. α satisfies (FOP)
2. There exists a sequence, $n_k > 0$, such that $d(\alpha^{n_k}, Inn(A)) \to 0$, as $k \to \infty$ (where d is any metric on $\mathrm{Aut}(A)$ which induces the point-norm topology).
3. The sequence $\{\alpha_*^n\}_{n \geq 0}$ has a convergent subsequence in $\mathrm{Aut}(A)_*$.

Proof (1) $\Rightarrow$ (2) Let $\{A_i\}$ be any increasing nest of finite dimensional subalgebras whose union is dense in A. It suffices to show that for each $i \in \mathbb{N}$ we can find integers n_i (with $n_i < n_{i+1}$) and unitaries $u_i \in A$ such that $\alpha^{n_i}|_{A_i} = Adu_i|_{A_i}$. So let $e_1, \ldots, e_{m_i}$ be minimal projections in A_i such that $\{[e_1], \ldots, [e_{m_i}]\}$ generate $K_0(A_i)$. Now, take an integer n_i such that $\alpha_*^{n_i}([e_j]) = [e_j]$, for $1 \leq j \leq m_i$ (note that n_i may be chosen larger than any specified number). Thus, the restriction of α^{n_i} to A_i agrees on K-theory with the natural inclusion map $A_i \hookrightarrow A$. Thus we may find the desired unitary , $u_i \in A$.

(2) $\Rightarrow$ (3) Clearly $\alpha_*^{n_i} \to id_{A*}$

(3) $\Rightarrow$ (4) Assume there exists a subsequence $\{m_k\}$ and an element $\beta_* \in \mathrm{Aut}(A)_*$ such that $\alpha_*^{m_k} \to \beta_*$. Then defining $n_k = m_k - m_{k-1}$ we have that

$$\alpha_*^{n_k} = \alpha_*^{m_k} \circ \alpha_*^{-m_{k-1}} \to \beta_* \circ \beta_*^{-1} = id_{A*}$$

Since $\Gamma(A)$ generates $K_0(A)$, it suffices to check (FOP) on this set. So, let $p \in A$ be any projection. Then let $\{a_i\}$ be any sequence which is dense in the unit ball of A, with $a_1 =$ p. Now recall that the metric on $\mathrm{Aut}(A)$ is defined as follows

$$d(\sigma, \gamma) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|\sigma(a_i) - \gamma(a_i)\|, \quad \text{for } \sigma, \gamma \in \mathrm{Aut}(A)$$

where choosing a different sequence just gives an equivalent metric (cf. [Ar]). Now, let $U_{1/2} = \{\sigma \in \mathrm{Aut}(A) : \text{there exists } \gamma \in Inn(A) \text{ such that } d(\sigma, \gamma) < 1/2\}$. Note that if $\sigma \in U_{1/2}$ then $\sigma_*([p]) = [p]$, since $a_1 = p$ and $\|\sigma(p) - \gamma(p)\| < 1$ for some $\gamma \in Inn(A)$.

Now, recall that by definition of the topology on $\mathrm{Aut}(A)_*$ (cf. [Po]) we have that $U_{1/2*} = \{\sigma_* : \sigma \in U_{1/2}\}$ is an open set in $\mathrm{Aut}(A)_*$ (obviously containing id_{A*}). Thus, there exists a $k \in \mathbb{N}$ large enough that $\alpha_*^{n_k} \in U_{1/2*}$. Thus, there exists $\sigma \in U_{1/2}$ with $\alpha_*^{n_k} = \sigma_*$ and hence $\alpha_*^{n_k}([p]) = \sigma_*([p]) = [p]$. $\square$

We will now show how to recover (the AF case of) a result of Pimsner on the AF embeddability of crossed products of commutative C*-algebras. (See [Pi])

Definition 4.5 If X is a compact, metrizable space and φ is a homeomorphism of X. Then $x \in X$ is called *pseudo-nonwandering* if for every $\epsilon > 0$, there exists a set of points $x_0, \ldots, x_{n+1}$ with $x = x_0 = x_{n+1}$ and $d(\varphi(x_i), x_{i+1}) < \epsilon$ for $0 \le i \le n$. The set of all pseudo-nonwandering points will be denoted by $X(\varphi)$.

Lemma 4.6(Pimsner) If $\varphi : X \to X$ is a homeomorphism of the compact metrizable space, X, then $x \notin X(\varphi) \Leftrightarrow$ there exists an open set, $U \subset X$ such that $\varphi(\overline{U}) \subset U$ and $x \in U \backslash \varphi(\overline{U})$.

As we are interested in the case when $C(X)$ is AF, we will assume from now on that X is also totally disconnected. Then we can find a basis of clopen sets, say $\{V_n\}$. We now claim that in this case, the open set U in Lemma 4.2 can be taken to be clopen. To see this, we assume that there exists an open set $U \subset X$ satisfying the two properties of the lemma. Now, since $\varphi(\overline{U})$ is a compact subset of U, we can find a finite subset of the clopen basis, say $V_{n_1}, \ldots, V_{n_k}$, such that

$$x \notin \cup_{i=1}^k V_{n_i}$$
$$\varphi(\overline{U}) \subset \cup_{i=1}^k V_{n_i} \subset U$$

Then letting $V = \varphi^{-1}(\cup_{i=1}^k V_{n_i})$, we get that V is a clopen set with the same properties as U.

Recall that when X is totally disconnected, $K_0(C(X)) = C(X, \mathbb{Z})$, the group of continuous functions from X to $\mathbb{Z}$, with positive cone given by the nonnegative functions.

Theorem 4.7(Pimsner) If $\varphi : X \to X$ is a homeomorphism of the compact, totally disconnected metric space X, then $C(X) \times_\varphi \mathbb{Z}$ is AF embeddable if and only if $X(\varphi) = X$, where the φ appearing in the crossed product denotes the corresponding automorphism of $C(X)$, i.e. $\varphi(f) = f \circ \varphi^{-1}$.

Proof ($\Rightarrow$) We prove the contrapositive. So assume that $X(\varphi) \neq X$. Then by Lemma 4.6 and the discussion which follows , we can find a clopen set, V, such that $\varphi(V)$ is properly contained in V. So, if $P \in C(X)$ is the projection with support V, then we have that $P \circ \varphi^{-1}$ is a projection in $C(X)$ which is dominated by P. Thus as elements in $C(X, \mathbb{Z})$ we have that $\varphi_*([P]) - [P]$ is a nonzero function in $-K_0^+(C(X))$ and hence $H_\varphi \cap K_0^+(C(X)) \neq 0$. Thus by Theorem 0.2 we have that $C(X) \times_\varphi \mathbb{Z}$ is not AF embeddable.

($\Leftarrow$) Assume that $X(\varphi) = X$. Now take $f \in C(X, \mathbb{Z})$ and assume that $f \circ \varphi^{-1} - f \geq 0$. Let $\{s_1, \ldots, s_k, -t_1, \ldots, -t_j\}$ be the range of f, where the $s_i,\ t_i$ are all nonnegative integers and $s_i > s_{i+1},\ t_l > t_{l+1}$ (and perhaps $s_k = 0$). Then define the clopen sets $E_i = f^{-1}(s_i),\ F_l = f^{-1}(t_l)$ and notice that these sets form a pairwise disjoint, finite clopen cover of X. Letting P_E denote the characteristic function of a clopen set, E, we can write

$$f = \textstyle\sum_{i=1}^k s_i P_{E_i} - \sum_{l=1}^j t_l P_{F_l}$$

and thus

$$f \circ \varphi^{-1} - f = \sum_{i=1}^k s_i P_{\varphi(E_i)} + \sum_{l=1}^j t_l P_{F_l} - \sum_{i=1}^k s_i P_{E_i} - \sum_{l=1}^j t_l P_{\varphi(F_j)}$$

Now, given $x \in E_1$ we have that the second summation above vanishes at x. Also, since $f \circ \varphi^{-1} - f \geq 0$, we have a unique index i such that

$$f \circ \varphi^{-1}(x) - f(x) = s_i - s_1 \geq 0$$

However, as $s_1 > s_i$, for $2 \leq i \leq k$, we conclude that $s_i = s_1$ and hence $\varphi(E_1) \supset E_1$. But this implies that $\varphi(E_1^c) \subset E_1^c$, where E^c denotes the complement. But then by hypothesis (and Lemma 4.2) we see that $\varphi(E_1) = E_1$. Repeating this argument we get that $\varphi(E_i) = E_i$ for $1 \leq i \leq k$. An obvious adaptation of this argument shows equality for the F_l's. Hence $f \circ \varphi^{-1} - f = 0$, i.e. $H_\alpha \cap K_0^+(C(X)) = 0$. $\Box$

Remark 4.8 We may now give a simple example showing that in general, (FOP) is not a necessary condition for AF embeddability.

Let X be the one point compactification of $\mathbb{Z}$ and define $\varphi : X \to X$ to be the homeomorphism taking $n \mapsto n+1$ and $\infty \mapsto \infty$. As Pimsner points out in [Pi], every point of X will be pseudo-nonwandering (and hence $C(X) \times_\varphi \mathbb{Z}$ will be AF embeddable) while it is not hard to see that every element of $K_0(C(X)) = C(X, \mathbb{Z})$ will have infinite orbit under the iterates of φ_*.

We now show that the existence of enough α_*-invariant states is a sufficient condition for AF embeddability. For convenience we will assume that A is unital and, again following the terminology in [Da], we define a *state* on $K_0(A)$ to be a contractive group homomorphism, τ: $K_0(A) \to (\mathbb{R}, \mathbb{R}_+, [\nvdash, \nVdash])$, with $\tau([1_A]) = 1$. There is a 1-1 correspondence between the states on $K_0(A)$ and the tracial states on the algebra, A (Theorem IV.5.3 in [Da]). Recall that a (tracial) state is said to be *faithful* if the only positive element in it's kernel is the zero element. We will denote by S_α the set of all α_*-*invariant* states, i.e. those states for which $\tau \circ \alpha_* = \tau$. Note that when A is unital, S_α is never empty (since $\mathbb{Z}$ is amenable).

The author would like to thank Professors L.G. Brown and N.C. Phillips for pointing out a substantial simplification in the proof of the following theorem.

Theorem 4.9 Let A be unital and $\alpha \in \mathrm{Aut}(A)$ be given. Assume that for every projection, $p \in A$, there exists a state, $\tau \in S_\alpha$, such that $\tau([p]) > 0$. Then $A \times_\alpha \mathbb{Z}$ is AF embeddable. In particular, if S_α contains a faithful state or if $A \times_\alpha \mathbb{Z}$ admits a faithful tracial state then it is AF embeddable.

Proof Again, we prove the contrapositive. So assume that there exists an element $x \in K_0(A)$ and a projection $q \in M_n(A)$ such that $\alpha_*(x) - x = [q] \neq 0$. Then clearly $\tau([q]) = 0$, for every $\tau \in S_\alpha$. But since $\Gamma(A)$ generates $K_0^+(A)$, we can find projections $p_1, \ldots, p_n \in A$ such that $[q] = [p_1] + \ldots + [p_n]$ and thus we see that $\tau([p_i]) = 0$ for every $\tau \in S_\alpha$ and $1 \leq i \leq n$. □

We are indebted to Kishimoto Sensei for pointing out the following corollary.

Corollary 4.10 If A is simple and unital then $A \times_\alpha \mathbb{Z}$ is AF embeddable for every $\alpha \in \mathrm{Aut}(A)$.

Proof This follows from Theorem 4.9 since every state on a simple dimension group is faithful.(cf. [Ef], Corollary 4.2) □

Remark 4.11 Corollary 4.10 shows the contrast between crossed products of unital and non-unital algebras. Indeed, it was first shown in [Cu] that the stabilizations of the Cuntz algebras, $\mathcal{O}_\backslash \otimes \mathbb{K}$, are isomorphic to crossed products of (non-unital) simple AF algebras. Thus crossed products of non-unital simple AF algebras can be purely infinite and hence not AF embeddable.

As a final application, we present a few more conditions which characterize AF embeddability.

Theorem 4.12 If A is an AF algebra (not necessarily unital) and $\alpha \in \mathrm{Aut}(A)$ then the following are equivalent

1. $A \times_\alpha \mathbb{Z}$ is AF embeddable
2. $\tilde{\mathrm{A}} \times_{\tilde{\alpha}} \mathbb{Z}$ is AF embeddable
3. For every $m \in \mathbb{Z}, \mathbb{A} \times_{\alpha^>} \mathbb{Z}$ is AF embeddable
4. There exists an integer, $m \neq 0$, such that $A \times_{\alpha^m} \mathbb{Z}$ is AF embeddable
5. There exists an AF algebra, B, and a group homomorphism $\theta : K_0(A \times_\alpha \mathbb{Z}) \to K_0(B)$ with $i)$ $\theta([p]) \in \Gamma(B)$ and $ii)$ $[p] \in Ker(\theta) \Rightarrow p = 0$, for every projection $p \in A$.

Proof (1) $\Leftrightarrow$ (2) It is easy to see that

$$K_0(\tilde{A}) = K_0(A) \oplus \mathbb{Z}$$
$$H_{\tilde{\alpha}} = H_\alpha \oplus 0$$

Thus it is clear that $H_{\tilde{\alpha}} \cap K_0^+(\tilde{A}) = 0 \Leftrightarrow H_\alpha \cap K_0^+(A) = 0$.

(1) $\Leftrightarrow$ (5) The hypotheses of (5) imply that $\theta \circ i_*$ is a faithful homomorphism (where $i : A \hookrightarrow A \times_\alpha \mathbb{Z}$ is the natural inclusion) with $\alpha_*(x) - x \in Ker(\theta \circ i_*)$, for all $x \in \Gamma(A)$. However, since $\Gamma(A)$ generates $K_0(A)$ we see that $H_\alpha \subset Ker(\theta \circ i_*)$. Thus the faithfulness of $\theta \circ i_*$ implies that $H_\alpha \cap K_0^+(A) = \{0\}$.

The converse is trivial.

By the equivalence of (1) and (2), we may assume for the remainder of the proof that A is unital.

(1)⇒(3) If $u \in A \times_\alpha \mathbb{Z}$ and $v \in A \times_{\alpha^m} \mathbb{Z}$ are the distinguished unitaries, then it is routine to check that the covariant representation

$$v \longmapsto u^m$$

$$a \longmapsto a$$

defines an embedding: $A \times_{\alpha^m} \mathbb{Z} \hookrightarrow A \times_\alpha \mathbb{Z}$.

(3) ⇒ (4) is immediate

(4) ⇒ (1) Assume $\varphi : A \times_{\alpha^m} \mathbb{Z} \longrightarrow B$ is an embedding into an AF algebra, B. Notice that in $K_0(B)$ we have that for every projection, $p \in A, [\varphi(p)] = [\varphi(upu^*)] = [\varphi(\alpha^m(p))]$. We will assume that m is positive, for if m is negative, it will be clear how to adapt the following argument.

Now, define $B_m = \oplus_0^{m-1} B$ and let $\beta \in \mathrm{Aut}(B_m)$ be the backwards cyclic permutation of the summands of B_m. That is, $\beta(b_0 \oplus \cdots \oplus b_{m-1}) = b_1 \oplus \cdots \oplus b_{m-1} \oplus b_0$. Now, identifying A with it's image in $A \times_{\alpha^m} \mathbb{Z}$, we define $\psi_i = \varphi|_A \circ \alpha^i, 0 \leq i \leq m-1$. Then we define $\psi = \oplus_0^{m-1} \psi_i : A \to B_m$, and it is clear that ψ is a *-monomorphism. It is also clear that β satisfies (FOP). Finally, using the fact that in $K_0(B)$ we know $[\varphi(p)] = [\varphi(\alpha^m(p))]$, one easily checks that $\beta_* \circ \psi_* = \psi_* \circ \alpha_*$. Thus the conclusion follows from Theorem 4.2 and Proposition 3.1. □

Remark 4.13 It follows from Lemma 2.8 in [Vo] that if $\mathcal{U} = \otimes_{\backslash \geq \infty} \mathcal{M}_\backslash$ is the Universal UHF algebra, then condition (5) implies that the crossed product can be embedded into $B \otimes \mathcal{U}$.

Remark 4.14 The author believes the equivalence of (1) and (2) to be a nontrivial application of Theorem 0.2 as he has been unable to find a direct proof of (1) ⇒ (2).

Remark 4.15 The equivalence of (1), (3) and (4) was proved in [Pi] when A is assumed to be any (not necessarily AF) unital, abelian C*-algebra.

Section 5: K-Theory

In this section we will show that our previous methods can be used to get rationally injective maps on $K_0(A \times_\alpha \mathbb{Z})$ (whenever $A \times_\alpha \mathbb{Z}$ is AF embeddable). That is, we will show that if $A \times_\alpha \mathbb{Z}$ is AF embeddable, then one can find a *-monomorphism, $\varphi : A \times_\alpha \mathbb{Z} \to \mathbb{B}$, where B is AF and $\varphi_* : K_0(A \times_\alpha \mathbb{Z}) \to \mathbb{K}_{\not\vdash}(\mathbb{B})$ is rationally injective. The author presented the main results of Sections 2, 3 and 4 in the Functional Analysis Seminar at Purdue University. He would like to thank Professor Larry Brown for asking the questions that led to this section of the paper.

The following definition is taken from [Vo].

Definition 5.1 If $\beta \in \mathrm{Aut}(B)$ then β is called *limit periodic* if there exists an increasing nest, $\{B_n\}$, of finite dimensional subalgebras (with $B = \overline{\bigcup B_n}$) and a sequence of positive integers, $d_n \in \mathbb{N}$, such that $\beta(B_n) = B_n$ and $\beta^{d_n}|_{B_n} = id_{B_n}$.

In Lemma 2.8 of [Vo] it is shown that crossed products of AF algebras by limit periodic automorphisms are AF embeddable. The following lemma may be well known to the experts, but the author is not aware of a specific reference and thus includes a proof for completeness.

Lemma 5.2 If B is unital and $\beta \in \overline{Inn(B)}$ is limit periodic (with respect to $\{B_n\}$ and with integers $\{d_n\}$) then the embedding of $B \times_\beta \mathbb{Z}$ into $B \otimes \mathcal{U}'$ (where $\mathcal{U}'$ is a UHF algebra) constructed in Lemma 2.8 of [Vo] induces an injective map on $K_0(B \times_\beta \mathbb{Z})$.

Sketch of Proof (cf. Lemma 2.8 in [Vo]). By the Pimsner-Voiculescu six term exact sequence, we have that $i_* : K_0(B) \to K_0(B \times_\beta \mathbb{Z})$ is an isomorphism, where $i : B \to B \times_\beta \mathbb{Z}$ is the natural inclusion map. Letting $\varphi : B \times_\beta \mathbb{Z} \to \mathbb{B} \otimes \mathcal{U}'$ be the embedding constructed in [Vo] we have the following commutative diagrams.

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[dr, "\varphi"] & B \times_\beta \mathbb{Z} \arrow[d, "\varphi"] \\ & B \otimes \mathcal{U}' \end{tikzcd}$$

$$\begin{tikzcd} K_0(B) \arrow[r, "i_*"] \arrow[dr, "\varphi_*"] & K_0(B \times_\beta \mathbb{Z}) \arrow[d, "\varphi_*"] \\ & K_0(B \otimes \mathcal{U}') \end{tikzcd}$$

Thus, it suffices to show that $\varphi_* : K_0(B) \to K_0(B \otimes \mathcal{U}')$ is an injective map. Now, as in the proof of Lemma 2.8 in [Vo], we define $U_n = M_{d_n}(\mathbb{C})$ and take maps $\varphi_n : B_n \to B_n \otimes U_n$ with commutativity in the diagram

$$\begin{tikzcd} B_n \arrow[r, "j_n"] \arrow[d, "\varphi_n"'] & B_{n+1} \arrow[d, "\varphi_{n+1}"'] \\ B_n \otimes U_n \arrow[r] & B_{n+1} \otimes U_{n+1} \end{tikzcd}$$

where we don't need to know what map the lower arrow is given by, j_n is the natural inclusion map and φ_n takes $b \mapsto \sum_{j=1}^{d_n} \beta^j(b) \otimes e_{j,j}$, where $e_{i,j}$ are the standard matrix units of M_{d_n}. Since φ is defined as the limit of the maps φ_n, we have the following commutative diagram

$$\begin{tikzcd} K_0(B_n) \arrow[r, "j_{n*}"] \arrow[d, "\varphi_{n*}"'] & K_0(B) \arrow[d, "\varphi_*"'] \\ K_0(B_n \otimes U_n) \arrow[r, "\Psi_{n*}"] & K_0(B \otimes \mathcal{U}') \end{tikzcd}$$

where $\Psi_n : B_n \otimes U_n \to B \otimes \mathcal{U}'$ is the induced map and actually, $\mathcal{U}' = \otimes_{\backslash\geq\infty}\mathcal{M}_{\lceil_\backslash}$. From this diagram we see that it suffices to show the following assertion.

Claim: If $x \in K_0(B_n)$ and $\varphi_n(x) = 0$ then $j_{n*}(x) = 0$.

So, assume we have $x = [p] - [q] \in K_0(B_n)$ (where $p,\ q$ are projections in the matrices over B_n) such that $\varphi_{n*}(x) = 0$. Then with the identification $K_0(B_n \otimes U_n) = K_0(M_{d_n}(B_n)) = K_0(B_n)$, we have that (in $K_0(B_n)$) $\varphi_n(x) = \sum_{j=1}^{d_n}[\beta^j(p)] - \sum_{j=1}^{d_n}[\beta^j(q)] = 0$. However, by assumption we have that in $K_0(B)$, $[r] = [\beta^j(r)]$, for every projection, r, in the matrices over B and for every $j \in \mathbb{Z}$. Thus, passing to $K_0(B)$ the above formula becomes $d_n([p] - [q]) = 0$ or $d_n(j_{n*}(x)) = 0$ and thus (since $K_0(B)$ is torsion free) $j_{n*}(x) = 0$. $\square$

We now present an analogue of the lemma of Spielberg (Lemma 3.4) which gives more control on K-theory. The main idea in the proof is similar to the proof of Lemma 1.14 is [Sp].

Lemma 5.3 If $(G,\ G^+)$ is a torsion free, ordered group then there exists a cone $\tilde{G}^+ \supset G^+$ such that $(G,\ \tilde{G}^+)$ is totally ordered and hence is a dimension group.

Proof Let $\mathcal{L} = \{\mathcal{E} \subset \mathcal{G} : \rangle\rangle\ \mathcal{E} \supset \mathcal{G}^+ \ \ \rangle\rangle\rangle\ \mathcal{E}$ is a semigroup $\ \rangle\rangle\rangle\rangle\ \mathcal{E} \cap (-\mathcal{E}) = \prime\ \}$ be partially ordered by inclusion. It is easy to check that the hypotheses of Zorn's Lemma are satisfied and thus we may choose a maximal element, $\tilde{G}^+ \in \mathcal{L}$. Now, choose $x \in G\backslash\tilde{G}^+$ and consider the semigroup $\{nx + e : n \in \mathbb{N},\ \ \in \tilde{\mathbb{G}}^+\}$. This semigroup clearly (properly) contains $\tilde{G}^+$, and thus by maximality we get that there exist nonnegative integers, $n,\ m \in \mathbb{N}$ (at least one of which is nonzero) , and elements, $e,\ f \in \tilde{G}^+$, such that $nx + e = -(mx + f) \neq 0$. Thus $(n+m)x \in -\tilde{G}^+$.

We now claim that $(G,\ \tilde{G}^+)$ is unperforated. So assume that we have found some nonzero element, $x \in G$, such that $kx \in \tilde{G}^+$ while $x \notin \tilde{G}^+$. Then the argument above (since $x \notin \tilde{G}^+$) provides us with a positive integer (letting $j = n + m$, from above) such that $jx \in -\tilde{G}^+$. Thus, $(kj)x \in \tilde{G}^+ \cap -\tilde{G}^+ = 0$. But this is a contradiction to the hypothesis that G is torsion free and hence $(G,\ \tilde{G}^+)$ is unperforated.

However, if we now take $x \in G\backslash\tilde{G}^+$ then we can find a positive integer j such that $jx \in -\tilde{G}^+$ which implies (since $(G,\ \tilde{G}^+)$ is unperforated) that $x \in -\tilde{G}^+$. Thus $(G,\ \tilde{G}^+)$ is totally ordered. $\square$

Remark 5.4 If $A\times_\alpha\mathbb{Z}$ is AF embeddable then it is easy to see that $K_0(A\times_\alpha\mathbb{Z}) = \mathbb{K}_{\not\vdash}(\mathbb{A})/\mathbb{H}_\alpha$ is an ordered group with the cone $K_0^+(A) + H_\alpha$. Clearly, the only thing that needs to be checked is that $K_0^+(A) + H_\alpha \cap -(K_0^+(A) + H_\alpha) = \{0\}$. However, this follows easily from the fact that $K_0^+(A) \cap H_\alpha = \{0\}$ by assumption (and Theorem 0.2).

Theorem 5.5 Assume that $A \times_\alpha \mathbb{Z}$ is AF embeddable and H is a subgroup of $K_0(A \times_\alpha \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{A})/\mathbb{H}_\alpha$ with the property that $H \cap (K_0^+(A) + H_\alpha) = \{0\}$ and $K_0(A \times_\alpha \mathbb{Z})/\mathbb{H}$ is torsion free. Then one may choose the AF embedding of $A \times_\alpha \mathbb{Z}$ such that the kernel of the induced map on $K_0(A \times_\alpha \mathbb{Z})$ is precisely H.

Proof To ease our notation somewhat, we begin by defining

$$G = K_0(A \times_\alpha \mathbb{Z})/\mathbb{H}$$
$$G^+ = (K_0^+(A) + H_\alpha) + H$$

where $K_0^+(A) + H_\alpha$ was shown to be a cone of $K_0(A \times_\alpha \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{A})/\mathbb{H}_\alpha$ in Remark 5.4 and thus G^+ is just the natural image of this cone in G. Now we claim that $(G,\ G^+)$ is an ordered group. It is clear that we have $G^+ + G^+ \subset G^+$ and $G = G^+ - G^+$ and thus we only have to check that $G^+ \cap -(G^+) = \{0\}$. So assume there are elements $x,\ y \in K_0^+(A)$ such that $(x + H_\alpha) + H = -(y + H_\alpha) + H$ (in G). This implies that $(x + y) + H_\alpha \in H \cap (K_0^+(A) + H_\alpha) = \{0\}$ (in $K_0(A \times_\alpha \mathbb{Z})$) and hence $x + y \in H_\alpha \cap K_0^+(A) = \{0\}$ (in $K_0(A)$). But, this implies that $x = y = 0$.

Thus we have that $(G,\ G^+)$ is a torsion free, ordered group and hence by Lemma 5.3 we can find a cone $\tilde{G}^+ \supset G^+$ and an AF algebra, B, such that $(K_0(B),\ K_0^+(B),\ \Gamma(B)) = (G,\ \tilde{G}^+,\ \tilde{G}^+)$. Note that if we let $\pi : K_0(A) \to K_0(B) = G$ be the canonical projection then π is a contractive, faithful group homomorphism. Hence, by Fact 1.1, there exists a *-monomorphism, $\varphi : A \to B$ with $\varphi_* = \pi$. The crucial properties in the remainder of the proof will be,

1. $\varphi : A \to B$ is a unital *-monomorphism
2. $K_0(B) = K_0(A \times_\alpha \mathbb{Z})/\mathbb{H}$
3. $Ker(\varphi_*) = Ker(\pi)$ where π is the composition of the canonical projection maps $K_0(A) \to K_0(A \times_\alpha \mathbb{Z}) \to \mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z})/\mathbb{H}$.

We now show that we may assume that A and B are unital and that $\varphi : A \to B$ is a unital injection. By the split exactness of the sequence

$$0 \to A \times_\alpha \mathbb{Z} \to \tilde{\mathbb{A}} \times_{\tilde{\alpha}} \mathbb{Z} \to \mathbb{C}(\mathbb{T}) \to \nvdash$$

we have that $K_0(\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z}) \oplus \mathbb{Z}$ and the image of H under this identification is $\tilde{H} = H \oplus 0 \subset K_0(\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z})$. We now claim that properties 1), 2) and 3) above still hold with the unitizations of $A,\ B$ and φ.

Now we recall that $K_0(\tilde{A}) = K_0(A) \oplus \mathbb{Z}$ and $K_0(\tilde{B}) = K_0(B) \oplus \mathbb{Z}$ and hence $ker(\tilde{\varphi}_*) = ker(\varphi_*) \oplus 0 \subset K_0(A) \oplus \mathbb{Z}$. Also, we observe that

$$K_0(\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z})/\tilde{\mathbb{H}} = (\mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z}) \oplus \mathbb{Z})/(\mathbb{H} \oplus \nvdash) = \mathbb{K}_{\nvdash}(\mathbb{B}) \oplus \mathbb{Z} = \mathbb{K}_{\nvdash}(\tilde{\mathbb{B}})$$

with all these identifications being natural. Thus properties 1) and 2) are satisfied. Finally, it is easy to see that the kernel of the composition of the projections maps

$$K_0(\tilde{A}) \to K_0(\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z}) \to \mathbb{K}_{\nvdash}(\tilde{\mathbb{A}} \times_{\tilde{\alpha}} \mathbb{Z})/\tilde{\mathbb{H}}$$

is precisely $Ker(\varphi_*) \oplus 0 = Ker(\tilde{\varphi}_*)$ since the above sequence is really just

$$K_0(A) \oplus 0 \to K_0(A \times_\alpha \mathbb{Z}) \oplus \mathbb{Z} \to (\mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z}) \oplus \mathbb{Z})/(\mathbb{H} \oplus \nvdash)$$

Thus it suffices to prove the theorem with $\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z}$ and $\tilde{H}$ and hence we may assume that A and B are unital and φ is a unit preserving, injective *-homomorphism with properties 1), 2) and 3) above.

Now, by the proof of Proposition 3.1 (and Proposition 2.8) we may further assume that, $\varphi : A \to B \otimes \mathcal{U}$, where $\mathcal{U}$ is the Universal UHF algebra and we have commutativity in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \otimes \mathcal{U} \\ {\scriptstyle Adv\circ\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle Adu\circ(id_B\otimes\sigma)} \\ A & \xrightarrow{\varphi} & B \otimes \mathcal{U} \end{array}$$

where $v \in A$, $u \in B \otimes \mathcal{U}$ are unitaries and $\sigma \in \mathrm{Aut}(\mathcal{U})$ is the automorphism (with the Rohlin property) from Example 2.2. Note that it was necessary to first arrange that φ be a unital map in order to appeal to Proposition 2.8 and that we have not changed the kernel of φ_* in doing so (although we now have that $K_0(B) = K_0(A \times_\alpha \mathbb{Z})/\mathbb{H}$ sits injectively inside $K_0(B \otimes \mathcal{U})$). Note also that $id_B \otimes \sigma$ is limit periodic.

Now, from the Pimsner-Voiculescu six term exact sequence we have that $K_0(B \otimes \mathcal{U}) = \mathcal{K}_\prime(\mathcal{B} \otimes \mathcal{U} \times_{\rangle\lceil_{\mathcal{B}}\otimes\sigma} \mathbb{Z})$. Thus commutativity in the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & A \times_{Adv\circ\alpha} \mathbb{Z} \\ {\scriptstyle \varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{\varphi}} \\ B \otimes \mathcal{U} & \xrightarrow{\qquad} & B \otimes \mathcal{U} \times_{\mathcal{A}\lceil\sqcap\circ(\rangle\lceil_{\mathcal{B}}\otimes\sigma)} \mathbb{Z} \end{array}$$

implies commutativity in the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\quad i_* \quad} & K_0(A \times_{Adv\circ\alpha} \mathbb{Z}) \\ {\scriptstyle \varphi_*}\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{\varphi}_*} \\ K_0(B \otimes \mathcal{U}) & \xrightarrow{\quad\cong\quad} & K_0(B \otimes \mathcal{U} \times_{\mathcal{A}\lceil\sqcap\circ(\rangle\lceil_{\mathcal{B}}\otimes\sigma)} \mathbb{Z}) \end{array}$$

Where the $\tilde{\varphi}$ on the right side of the first diagram is now the natural extension of φ to the crossed products (i.e. it no longer denotes the unital extension) and i is the natural inclusion map.

Finally, since $Ker(\varphi_*) = Ker(\tilde{\varphi}_* \circ i_*) = Ker(\pi)$ where π was the composition of the maps

$$K_0(A) \to K_0(A \times_\alpha \mathbb{Z}) \to \mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z})/\mathbb{H} = \mathbb{K}_{\nvdash}(\mathbb{B}) \hookrightarrow \mathbb{K}_{\nvdash}(\mathbb{B} \otimes \mathcal{U})$$

it is now a routine exercise to verify that $Ker(\tilde{\varphi}_*) = H$ (under the identifications $K_0(A \times_{Adv\circ\alpha} \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z})$ and $K_0(B \otimes \mathcal{U} \times_{\mathcal{A}\lceil\sqcap(\circ\rangle\lceil_{\mathcal{B}}\otimes\sigma)} \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{B} \otimes \mathcal{U} \times_{\rangle\lceil_{\mathcal{B}}\otimes\sigma} \mathbb{Z})$). Thus the proof is complete since $id_B \otimes \sigma$ is limit periodic and hence from Lemma 5.2 we have that $K_0(B \otimes \mathcal{U} \times_{\rangle\lceil_{\mathcal{B}}\otimes\sigma} \mathbb{Z})$ gets embedded into $K_0(B \otimes \mathcal{U} \otimes \mathcal{U}')$ where $\mathcal{U}'$ is the UHF algebra constructed in Lemma 5.2. □

Corollary 5.6 If $A \times_\alpha \mathbb{Z}$ is AF embeddable, then one may choose an embedding which induces a rationally injective map on $K_0(A \times_\alpha \mathbb{Z})$.

Proof Let $T(A \times_\alpha \mathbb{Z}) \subset \mathbb{K}_{\nvdash}(\mathbb{A} \times_\alpha \mathbb{Z}) = \mathbb{K}_{\nvdash}(\mathbb{A})/\mathbb{H}_\alpha$ denote the torsion subgroup. To apply Theorem 5.5 we only need to see that $T(A \times_\alpha \mathbb{Z}) \cap (\mathbb{K}_{\nvdash}^+(\mathbb{A}) + \mathbb{H}_\alpha) = \{\nvdash\}$. But this follows easily from the fact that $K_0^+(A) \cap H_\alpha = \{0\}$. □

Remark 5.7 It is easy to prove that if $K_0(A)$ is a divisible group then $K_0(A \times_\alpha \mathbb{Z})$ is always torsion free. However, it not hard to construct an example where $A \times_\alpha \mathbb{Z}$ is AF embeddable and $K_0(A \times_\alpha \mathbb{Z})$ has torsion.

Remark 5.8 It is clear that, in general, our constructions will not yield isomorphisms on K-theory since we must tensor with some UHF algebra to get the desired embeddings.

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